NUMEROUS SERIES (I - PART)

Consider a number series $a_1 + a_2 + a_3 + ... + a_n + ... = \sum_{n=1}^{\infty} a_n$ with positive members.

The sum is $S_n = a_1 + a_2 + a_3 + \ldots + a_n = \sum_{k=1}^{n} a_k$, and we call it **partial sum**.

We are looking for $\lim_{n\to\infty} S_n$.

If $\lim_{n\to\infty} S_n$ =S (number) then series **converges**, and if $\lim_{n\to\infty} S_n = \pm\infty$ or does not exist, then the series **diverges.**

Partial sums are in fact:

$$
\begin{aligned}
&S_1 = a_1\\
&S_2 = a_1 + a_2\\
&S_3 = a_1 + a_2 + a_3\\
&...\\
&S_n = a_1 + a_2 + a_3 + ...a_n\\
&...
\end{aligned}
$$

Example 1.

For a given series $\frac{1}{1\cdot 4} + \frac{1}{4\cdot 7} + ... + \frac{1}{(3n-2)\cdot(3n+1)} + ...$ determine S_n and found $\lim_{n\to\infty} S_n$.

Solution:

$$S_n = \frac{1}{1\cdot 4} + \frac{1}{4\cdot 7} + ... + \frac{1}{(3n-2)\cdot(3n+1)}$$

Decomposes a rational function on FACTORS:

$$
\begin{aligned}
&\frac{1}{(3n-2)\cdot(3n+1)} = \frac{A}{3n-2} + \frac{B}{3n+1} ..../*(3n-2)\cdot(3n+1)\\
&1 = A(3n+1) + B(3n-2)\\
&1 = 3An + A + 3Bn - 2B\\
&1 = (3A+3B)n + A - 2B ....\text{Compares:}
\end{aligned}
$$

$3A+3B=0$

$\underline{A-2B=1 .... /*(-3)}$

$3A+3B=0$

$\underline{-3A+6B=-3}$

$9B=-3 \rightarrow \boxed{B=-\frac{1}{3}} \rightarrow \boxed{A=\frac{1}{3}}$

$$\frac{1}{(3n-2)\cdot(3n+1)}=\frac{\frac{1}{3}}{3n-2}+\frac{-\frac{1}{3}}{3n+1}=\frac{1}{3}(\frac{1}{3n-2}-\frac{1}{3n+1})$$

Now we return to the task and this applies to each addend of the series:

$$S_n=\frac{1}{3}[(1-\frac{1}{4})+(\frac{1}{4}-\frac{1}{7})+(\frac{1}{7}-\frac{1}{10})+....+(\frac{1}{3n-5}-\frac{1}{3n-2})+(\frac{1}{3n-2}-\frac{1}{3n+1})]$$

$$S_n=\frac{1}{3}[1-\frac{1}{4}+\frac{1}{4}-\frac{1}{7}+\frac{1}{7}-\frac{1}{10}+....+\frac{1}{3n-5}-\frac{1}{3n-2}+\frac{1}{3n-2}-\frac{1}{3n+1})]$$

$$S_n=\frac{1}{3}[1-\frac{1}{3n+1})]$$

$$\lim_{n\to\infty} S_n=\lim_{n\to\infty}\frac{1}{3}[1-\frac{1}{3n+1})]=\frac{1}{3}\lim_{n\to\infty}[1-\underbrace{\boxed{\frac{1}{3n+1}}}_{teži\ 0})]=\frac{1}{3}$$

$\boxed{S=\frac{1}{3}}$

<u>Example 2.</u>

Examine the convergence of series $\sum_{n=1}^{\infty}\frac{2n-1}{3n+2}$

<u>Solution:</u>

Here is our job easy! Theorem is true:

If the series $\sum_{n=1}^{\infty} a_n$ converges , then $\lim_{n\to\infty} a_n$=0, if $\lim_{n\to\infty} a_n \neq 0$ then line certainly does not converge.

Therefore, we ask $\lim_{n\to\infty}\frac{2n-1}{3n+2}$ and we know that $\lim_{n\to\infty}\frac{2n-1}{3n+2}=\frac{2}{3}\neq 0$, we are confident that this series diverges.

Example 3.

Examine the convergence of series: $\sum_{n=1}^{\infty}\frac{1}{2n-1}$

Solution:

First we will examine: $\lim_{n\to\infty}\frac{1}{2n-1}=\frac{1}{\infty}=0$. Does this mean that series converges? **NO!**

It may converge or diverges , the theorem helps in a situation when we get some number $\neq 0$ for the solution, and then we are sure that the series diverges. Thus we have to investigate further…

$$S_{2^1}=S_2=1+\frac{1}{3}$$

$$S_{2^2}=S_4=1+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}$$

$\ldots\ldots.etc.$

$$S_{2^m}=\frac{1}{1}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+\frac{1}{11}+\frac{1}{13}+\ldots\ldots+\frac{1}{2\cdot 2^m-1}$$

then:

$$S_{2^m}=(\frac{1}{1}+\frac{1}{3})+(\frac{1}{5}+\frac{1}{7})+(\frac{1}{9}+\frac{1}{11}+\frac{1}{13}+\frac{1}{15})\ldots\ldots+(\frac{1}{2^m+1}+\frac{1}{2^m+3}+\ldots+\frac{1}{2\cdot 2^m-1})$$

It is obvious that:

$$\frac{1}{1}+\frac{1}{3}>1$$

$$\frac{1}{5}+\frac{1}{7}>\frac{1}{8}+\frac{1}{8}=\frac{1}{4}$$

$$\frac{1}{9}+\frac{1}{11}+\frac{1}{13}+\frac{1}{15}>\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}=\frac{1}{4}$$

$\ldots\ldots.$

$$\frac{1}{2^m+1}+\frac{1}{2^m+3}+\ldots+\frac{1}{2\cdot 2^m-1}>\frac{2^{m-1}}{2^{m+1}}=\frac{1}{4}$$

we have:

$$S_{2^m}>1+\frac{m-1}{4}$$

As is $\lim_{m\to\infty}(1+\frac{m-1}{4})=\infty$, this tells us that a series **diverges!**

Example 4.

Examine the convergence of series: $\sum_{n=1}^{\infty} \frac{1}{n}$.

Solution:

And here is a similar situation as just: $\lim_{n\to\infty} \frac{1}{n} = \frac{1}{\infty} = 0$.

We use a similar trick as in the previous task…

We are looking at a series of partial sums $S_k, k \in N$ that is always growing. His subsequence $S_{2^m}, m \in N$ is:

$$S_{2^1} = S_2 = 1 + \frac{1}{2}$$

$$S_{2^2} = S_4 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}$$

.......*etc.*

$$S_{2^m} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + ... + \frac{1}{2^{m-1}+1} + \frac{1}{2^{m-1}+2} + \cdots + \frac{1}{2^m}$$

Group members:

$$S_{2^m} = \frac{1}{1} + \frac{1}{2} + (\frac{1}{3} + \frac{1}{4}) + (\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}) + ... + (\frac{1}{2^{m-1}+1} + \frac{1}{2^{m-1}+2} + \cdots + \frac{1}{2^m})$$

$$\frac{1}{3} + \frac{1}{4} > \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

$$\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{1}{2}$$

....

$$\frac{1}{2^{m-1}+1} + \frac{1}{2^{m-1}+2} + \cdots + \frac{1}{2^m} > \frac{1}{2^m} + \frac{1}{2^m} + \cdots + \frac{1}{2^m} = \frac{1}{2}$$

We have:

$$S_{2^m} > 1 + \frac{m}{2}$$

Here is $\lim_{m\to\infty}(1 + \frac{m}{2}) = \infty$, and series **diverges.**

Cauchy criteria (test)

Necessary and sufficient condition for $\sum_{n=1}^{\infty} a_n$ to converges is that for arbitrary $\varepsilon > 0$, there is a natural number $N = N(\varepsilon)$ so that for $n > 0 \wedge p > 0$ is true : $\left|S_{n+p} - S_n\right| < \varepsilon$

Example 5.

Using the Cauchy criteria , prove that series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

Solution:

In the previous example we prove that this series diverges. Now our job is to prove that using the Cauchy test.

Consider that p = n. Then we have:

$$\left|S_{n+n} - S_n\right| = \left|S_{2n} - S_n\right| = \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + ... + \frac{1}{2n}$$

As is :

$$\frac{1}{n+1} > \frac{1}{2n}$$
$$\frac{1}{n+2} > \frac{1}{2n}$$
$$\frac{1}{n+3} > \frac{1}{2n}$$

....

This back up and get:

$$\left|S_{n+n} - S_n\right| = \left|S_{2n} - S_n\right| = \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + ... + \frac{1}{2n} > n \cdot \frac{1}{2n} = \frac{1}{2}$$

If we initially take to, say $\varepsilon = \frac{1}{4}$ and we have that $\varepsilon > \frac{1}{2}$, We can conclude that the series diverges.

Example 6.

Using the Cauchy criteria , prove that series $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n(n+1)}}$ diverges.

Solution:

Take that $\varepsilon=\frac{1}{4}$ **and p = n. We have:**

$$|S_{n+n}-S_n|=|S_{2n}-S_n|=\frac{1}{\sqrt{(n+1)(n+2)}}+\frac{1}{\sqrt{(n+2)(n+3)}}+.......+\frac{1}{\sqrt{2n(2n+1)}}$$

Now think:

$$(n+1)(n+2)<(n+2)^2\rightarrow\frac{1}{(n+1)(n+2)}<\frac{1}{(n+2)^2}\rightarrow\frac{1}{\sqrt{(n+1)(n+2)}}<\frac{1}{\sqrt{(n+2)^2}}\rightarrow\boxed{\frac{1}{\sqrt{(n+1)(n+2)}}<\frac{1}{n+2}}$$

$$(n+2)(n+3)<(n+3)^2\rightarrow\frac{1}{(n+2)(n+3)}<\frac{1}{(n+3)^2}\rightarrow\frac{1}{\sqrt{(n+2)(n+3)}}<\frac{1}{\sqrt{(n+3)^2}}\rightarrow\boxed{\frac{1}{\sqrt{(n+2)(n+3)}}<\frac{1}{n+3}}$$

....

And so on.

Now, we have :

$$|S_{n+n}-S_n|=|S_{2n}-S_n|=\frac{1}{\sqrt{(n+1)(n+2)}}+\frac{1}{\sqrt{(n+2)(n+3)}}+.......+\frac{1}{\sqrt{2n(2n+1)}}>$$

$$>\frac{1}{n+2}+\frac{1}{n+3}+...+\frac{1}{2n+1}>\frac{1}{4}$$

We prove that this series diverges.

Comparable criteria:

Valid for two series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$

i) If $a_n < b_n$ then a) $\sum_{n=1}^{\infty} b_n$ convergent $\Rightarrow$ $\sum_{n=1}^{\infty} a_n$ convergent

b) $\sum_{n=1}^{\infty} a_n$ divergent $\Rightarrow$ $\sum_{n=1}^{\infty} b_n$ divergent

ii) If $\frac{a_{n+1}}{a_n} < \frac{b_{n+1}}{b_n}$ then a) $\sum_{n=1}^{\infty} b_n$ convergent $\Rightarrow$ $\sum_{n=1}^{\infty} a_n$ convergent

b) $\sum_{n=1}^{\infty} a_n$ divergent $\Rightarrow$ $\sum_{n=1}^{\infty} b_n$ divergent

iii) If $\lim_{n\to\infty} \frac{a_n}{b_n} = M$, ($M \neq 0$ and M is a finite number) series simultaneously are convergent or divergent

Most often series we used for comparison is $\sum_{n=1}^{\infty} \frac{1}{n^k}$; for k>1series is convergent , for $k \leq 1$ is divergent.

Example 7.

Examine the convergence of series: $\sum_{n=1}^{\infty} \frac{1}{2n-1}$

Solution:

Think:

$2n-1 > n$ starting from n=2, so:

$\frac{1}{2n-1} < \frac{1}{n}$

As the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges , series $\sum_{n=1}^{\infty} \frac{1}{2n-1}$ diverges by the comparative criteria

Some teachers do this directly:

$\frac{1}{2n-1} \sim \frac{1}{2n}$ when $n \to \infty$

Then is $\sum_{n=1}^{\infty} \frac{1}{2n} = \frac{1}{2} \boxed{\sum_{n=1}^{\infty} \frac{1}{n}}$, and conclude that a given series diverges.

Example 8.

Examine the convergence of series: $\sum_{n=1}^{\infty}\frac{1}{n\sqrt{n+1}}$

Solution:

We will use the faster way, and you of course work to your professor requires …

$$\frac{1}{n\sqrt{n+1}} \sim \frac{1}{n(n+1)^{\frac{1}{2}}} \sim \frac{1}{n^1 \cdot n^{\frac{1}{2}}} = \frac{1}{n^{\frac{3}{2}}} \quad \text{when} \quad n \to \infty .$$

Mark $\sim$ mean that these expressions behave similarly when $n \to \infty$

Series $\sum_{n=1}^{\infty}\frac{1}{n^{\frac{3}{2}}}$ converges , so converges and order $\sum_{n=1}^{\infty}\frac{1}{n\sqrt{n+1}}$.

Of course, we could have used comparable criteria, where $\sum_{n=1}^{\infty}\frac{1}{n^2}$ is for comparison.

Example 9.

Examine the convergence of series: $\sum_{n=1}^{\infty}\frac{\sqrt{n+1}-\sqrt{n}}{\sqrt[3]{n}}$

Solution:

Here we first make a rationalization:

$$\sum_{n=1}^{\infty}\frac{\sqrt{n+1}-\sqrt{n}}{\sqrt[3]{n}} \cdot \frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = \sum_{n=1}^{\infty}\frac{(\sqrt{n+1})^2-(\sqrt{n})^2}{\sqrt[3]{n}\cdot(\sqrt{n+1}+\sqrt{n})} = \sum_{n=1}^{\infty}\frac{n+1-n}{\sqrt[3]{n}\cdot(\sqrt{n+1}+\sqrt{n})} =$$
$$= \sum_{n=1}^{\infty}\frac{1}{\sqrt[3]{n}\cdot(\sqrt{n+1}+\sqrt{n})}$$

Now, think:

$$\frac{1}{\sqrt[3]{n}\cdot(\sqrt{n+1}+\sqrt{n})} \sim \frac{1}{\sqrt[3]{n}\cdot(\sqrt{n}+\sqrt{n})} = \frac{1}{n^{\frac{1}{3}}\cdot 2n^{\frac{1}{2}}} = \boxed{\frac{1}{2n^{\frac{5}{6}}}} \quad \text{when } n \to \infty$$

So, we have $\sum_{n=1}^{\infty}\frac{1}{2n^{\frac{5}{6}}} = \frac{1}{2}\boxed{\sum_{n=1}^{\infty}\frac{1}{n^{\frac{5}{6}}}}$, this series diverges and series $\sum_{n=1}^{\infty}\frac{\sqrt{n+1}-\sqrt{n}}{\sqrt[3]{n}}$ diverges to.

Example 10.

Examine the convergence of series: $\sum_{n=1}^{\infty} \sin\frac{\alpha}{n}$

Solution:

Here we use $\sin\frac{\alpha}{n} \sim \frac{\alpha}{n}$ when $n \to \infty$

We have : $\sum_{n=1}^{\infty} \frac{\alpha}{n} = \alpha \cdot \boxed{\sum_{n=1}^{\infty} \frac{1}{n}}$ diverges, so $\sum_{n=1}^{\infty} \sin\frac{\alpha}{n}$ diverges.

Example 11.

Examine the convergence of series $\sum_{n=1}^{\infty} \frac{n \sin\frac{1}{n}}{\sqrt{n^2+1}}$

Solution:

When $n \to \infty$ there are $\sin\frac{1}{n} \sim \frac{1}{n}$ and $\sqrt{n^2+1} \sim \sqrt{n^2} \sim n$,so, we have :

$$\frac{n \sin\frac{1}{n}}{\sqrt{n^2+1}} \sim \frac{n \cdot \frac{1}{n}}{n} = \frac{1}{n}$$

$\sum_{n=1}^{\infty} \frac{1}{n}$ diverges , and $\sum_{n=1}^{\infty} \frac{n \sin\frac{1}{n}}{\sqrt{n^2+1}}$ diverges .

In the next file we continue with the criteria for convergence ...